Bootstrap in separable Hilbert spaces with applications to functional data analysis

Gil Gonzalez-Rodriguez

CMStatistics
Computational and Methodological Statistics

CMStatistics - Mieres

Departament of Statistics and O.R.
University of Oviedo

Oviedo, August 2016
Overview

• Consistent bootstrap approaches in separable Hilbert spaces will be derived from the well-known powerful results for general empirical process

• Results on naive bootstrap, bootstrap with arbitrary sample size, wild bootstrap, several weighted bootstrap methods etc will be obtained

• An illustration to show how to employ the approach in the context of a functional regression problem will be detailed

• Empirical results involving an ANOVA problem for fuzzy data will also be considered
Bootstrap for empirical processes
Empirical processes

- \((\Omega, \mathcal{A}, P_r)\) — probability space, \(X : \Omega \rightarrow \mathcal{X}\) — random element, \(P\) — induced probability on \(\mathcal{X}\), \(X_1, \ldots, X_n\) iid

- **Empirical measure:**

\[
P_n(\omega) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(\omega)
\]

\(\delta_x : \mathcal{X} \rightarrow \mathbb{R}\) assigning 1 to \(x\) and 0 otherwise

- For any measurable function \(f : \mathcal{X} \rightarrow \mathbb{R}\)

\[
P f = \int f dP \quad \text{and} \quad P_n f = \int f dP_n = \frac{1}{n} \sum_{i=1}^{n} f(X_i)
\]

- **Empirical process:** \(\{P_n f, f \in \mathcal{F}\}\), where \(\mathcal{F}\) is any class of measurable functions \(f : \mathcal{X} \rightarrow \mathbb{R}\)
Example

- **Empirical distribution function**
  
  \[ \mathcal{X} = \mathbb{R} \]
  
  \[ \mathcal{F} = \{ I_{x \leq t}, t \in \mathbb{R} \} \]
  
  \[ F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I_{X_i \leq t} \]
  
  is the empirical process in \( \{ P_n f, f \in \mathcal{F} \} \)

Basic reference for empirical processes

Notation

- $\|g\|_\mathcal{F} = \sup_{f \in \mathcal{F}} |g(f)|$
- $l^\infty(\mathcal{F}) = \{g : \mathcal{F} \to \mathbb{R}, \|g\|_\mathcal{F} < \infty\}$ is a Banach space
  ✓ Sum and product by a scalar pointwise-defined
- $G_P = \{G_P(f), f \in \mathcal{F}\}$ will denote a centered Gaussian process indexed by $\mathcal{F}$ with covariance
  \[ EG_P(f_1)G_P(f_2) = \int f_1 f_2 dP - \int f_1 dP \int f_2 dP \]
  ✓ $\gamma_P$ will denote its associated measure

Central Limit Theorem classes

- $\mathcal{F} \in CLT(P) \iff \{n^{1/2}(\mathbb{P}_n - P)(f), f \in \mathcal{F}\}$ converges weakly in $l^\infty(\mathcal{F})$ to a Radon centered Gaussian probability measure $\gamma_P$ over $l^\infty(\mathcal{F})$
Bootstrapping general empirical measures


- Let $F = \sup_{f \in \mathcal{F}} |f|$ be so that $F(x) < \infty \ \forall x \in \mathcal{X}$ and $\mathcal{F}$ is image admissible Suslin

- Let $\{X_{n,j}^\omega\}_{j=1,...,n}$ be i.i.d.-$\mathbb{P}_n(\omega)$ and let $\mathbb{P}_n^*(\omega)$ be the empirical measure based on $\{X_{n,j}^\omega\}_{j=1,...,n}$

- **Theorem:** $a)$ and $b)$ are equivalent:
  
  $a)$ $\int F^2 d\mathbb{P} < \infty$ and $\mathcal{F} \in CLT(\mathbb{P})$
  
  $b)$ There exists a centered Gaussian process indexed by $\mathcal{F}$ whose law is Radon on $l^\infty(\mathcal{F})$ so that,

  $$n^{1/2}(\mathbb{P}_n^*(\omega) - \mathbb{P}_n(\omega)) \to G$$

  weakly in $l^\infty(\mathcal{F}) \ P$-a.s.

  If $a)$ or $b)$ are fulfilled, then $G = G_P$
Bootstrap in separable Banach spaces

- **Corollary:** If $X_i$ are iid-$X$ random elements on a separable Banach space $B$, then

$$E\|X\|^2 < \infty \text{ and } X \in CLT$$

$$\iff \sum_{j=1}^{n} (X_{nj}^* - \overline{X}_n)/n^{1/2} \to G_X$$

weakly $P$-a.s.
Exchangeably weighted bootstraps of empirical processes


**Theorem** Let \( \{W_{nj}^*\}_{j=1}^n \) be an array of random variables so that

A1) \((W_{n1}^*, \ldots, W_{nn}^*)\) are exchangeable

A2) \(W_{nj}^* \geq 0 \ \forall j\) and \(\sum_{j=1}^n W_{nj}^* = n \ \forall n \in \mathbb{N}\)

A3) \(\sup_n \int_0^\infty (P(|W_{n1}^*| > t))^{1/2} dt < \infty\)

A4) \(\lim_{\lambda \to 0} \limsup_n \sup_{t \geq \lambda} t^2 P(W_{n1}^* \geq t) = 0\)

A5) \((1/n) \sum_{j=1}^n (W_{nj}^* - 1)^2 \to c^2 > 0\) in probability

If \(\mathcal{F} \in M(P)\), \(\int F^2 dP < \infty\) and \(\mathcal{F} \in CTL(P)\) then

\[
n^{1/2} \left( \frac{1}{n} \sum_{j=1}^n W_{nj}^* \delta_{X_j^\omega} - P_n \right) \to c \ G_P
\]

weakly in \(l^\infty(\mathcal{F})\) \(P\)-a.s.
Exchangeably weighted bootstraps with arbitrary sample size

- **Corollary:** Let \( \{X_{n,j}^\omega\}_{j=1,...,m} \) be i.i.d.-\( \mathbb{P}_n(\omega) \) with associated empirical measure \( \mathbb{P}_{m,n}^*(\omega) \).

If \( \mathcal{F} \in M(P) \), \( \int F^2 dP < \infty \) and \( \mathcal{F} \in CTL(P) \), then if \( m \wedge n \to \infty \),

\[
m^{1/2}(\mathbb{P}_{m,n}^*(\omega) - \mathbb{P}_n(\omega)) \to G_P
\]

weakly in \( l^\infty(\mathcal{F}) \) \( P \)-a.s.
Bootstrap in Hilbert spaces
Separable Hilbert spaces

- \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) : separable Hilbert space with norm \(\| \cdot \|\)
- Let \(X : \Omega \to \mathcal{H}\) be measurable and so that \(E\|X\|^2 < \infty\)

Class of index functions

- Let \(\mathcal{F} = \{f \in \mathcal{H}'|\|f\|' \leq 1\}\)
  - \(F(h) = \sup_{f \in \mathcal{F}} |f(h)| = \|h\| < \infty\) for all \(h \in \mathcal{H}\)
  - \(\int F^2dP = E(\|X\|^2) < \infty\)
  - \(\mathcal{F}\) is image admissible Suslin
Linkage mapping

Let \( D : \mathcal{H} \to l^\infty(\mathcal{F}) \) be so that \( D(h)(f) = D_h(f) = f(h) \) for all \( h \in \mathcal{H} \) and all \( f \in \mathcal{F} \)

- \( D \) is a bounded and linear (so continuous) operator with range \( R(D) \subset l^\infty(\mathcal{F}) \)
- \( \|D(h)\|_{\mathcal{F}} = \|h\| \) for all \( h \in \mathcal{H} \), then there exists \( D^{-1} : R(D) \to \mathcal{H} \) so that \( D^{-1} \) is continuous
- \( R(D) \) is closed \( \Rightarrow \) there exists a continuous extension \( D^{-1} : l^\infty(\mathcal{F}) \to \mathcal{H} \)

Linkage between processes

Let \( \{X_i\}_{i=1}^n \) be iid-\( X \) and \( \{X_{nj}\}_{j=1,...,n} \) iid-\( F_n(\omega) \),

- \( D \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - E(X)) \right) = n^{1/2}(\mathbb{P}_n - P) \)
- \( D \left( \frac{1}{\sqrt{n}} \sum_{j=1}^n (X_{nj}^* - X_n) \right) = n^{1/2}(\mathbb{P}_n^*(\omega) - \mathbb{P}_n(\omega)) \)
\( \mathcal{F} \) is a Central Limit Theorem Class I

- **CLT in \( H \):** Let \( \{X_i\}_{i=1}^n \) be iid random elements in \( H \) with \( E(\|X\|^2) < \infty \) \( \Rightarrow \)

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - E(X)) \rightarrow Z_X
\]

weakly in \( \mathcal{H} \), where \( Z_X \) is a centered Gaussian random element on \( \mathcal{H} \) so that

\[
E(Z_X \langle h, Z_X \rangle) = EX \langle h, X \rangle - EX \langle h, EX \rangle
\]

for all \( h \in \mathcal{H} \)
\( \mathcal{F} \) is a Central Limit Theorem Class II

- By applying the continuous operator \( D \),

\[
 n^{1/2} (P_n - P) = D \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - E(X)) \right) \rightarrow D(Z_X)
\]

weakly on \( l^\infty(\mathcal{F}) \), where \( D(Z_X) \) is a Radon centered Gaussian process so that

\[
ED(Z_X)(f_1)D(Z_X)(f_2) = Ef_1(X)f_2(X) - Ef_1(X)Ef_2(X)
\]

\( \Rightarrow \mathcal{F} \in CLT(P) \)
Main consequences

Exchangeable Weighted Bootstrap

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} (W_{nj}^* X_j^\omega - \overline{X}_n) \to cZ_X \text{ weakly in } \mathcal{H} \quad P - c.s.$$ 

“Naive” (multinomial) Bootstrap

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} (X_{nj}^\omega - \overline{X}_n) \to Z_X \text{ weakly in } \mathcal{H} \quad P - c.s.$$ 

Bootstrap with arbitrary sample size

$$\frac{1}{\sqrt{m}} \sum_{j=1}^{m} (X_{mj}^\omega - \overline{X}_n) \to Z_X \text{ weakly in } \mathcal{H} \quad P - c.s.$$
iid-weighted bootstraps

Let \( \{Y_j\}_{j \in \mathbb{N}} \) be iid positive random variables with

\[
\int_0^\infty (P(|Y_1| > t))^{1/2} dt < \infty
\]

and define bootstrap weights by

\[
W_{nj}^* = \frac{Y_j}{Y_j^*}
\]

- "Smooother" bootstraps than the multinomial one
- \( c^2 = \frac{Var(Y_1)}{E(Y_1^2)} \)
- Bayesian bootstrap: \( Y_1 \sim exp(1) \)

Double bootstrap

Draw \( \{X_{i}^{**}\}_{i=1}^{n} \) at random with replacement from \( \{X_{i}^*\}_{i=1}^{n} \).

- More concentrated on few data points than the multinomial one
- \((W_{n1}, \ldots, W_{nn}) \sim Mult_n(n, (M_{n1}/n, \ldots, M_{nn}/n))\)
  where \( M_n \sim Mult_n(n, (1/n, \ldots, 1/n)) \)
- \( c^2 = 2 \)
Urn model bootstraps

Put $K > 0$ copies of each observed data in an urn

- Weights as multivariate hypergeometric distribution
- Polya-Eggenberger bootstrap

Bootstrap generated by deterministic weights

Let $\{V_{nj}\}_{j=1}^{n}$ ($n \in \mathbb{N}$) be an array of nonnegative numbers so that $\sum_{j=1}^{n} V_{nj} = n \forall n \in \mathbb{N}$. Let $R_n$ be a random permutation uniformly distributed on $\Pi_n$. Define $W_{nj}^* = V_{nR_n(j)} \forall j = 1, \ldots, n$.

- Check whether conditions A3-A5 are satisfied.
- Particular case: delete-h jackknife

$V_{nj} = n/(n - h_n)$ \quad (j = 1, \ldots, n - h_n)

$V_{nj} = 0$ \quad (j = n - h_n + 1, \ldots, n).

$V_{nj} = n/(n - h_n)$ \quad (j = 1, \ldots, n - h_n)

$V_{nj} = 0$ \quad (j = n - h_n + 1, \ldots, n).

If $h_n/n \rightarrow \alpha \in (0, 1)$ then A3-A5 are satisfied with $c^2 = \alpha/(1 - \alpha)$. 

Additional tools - Wild bootstrap


- **Theorem:** Let \( \{g_i^*\}_{i=1}^n \) be a sequence of i.i.d. random variables with \( \int_0^\infty (P(\|g_1^*\| > t)^{1/2})dt < \infty \), \( E(g_1^*) = 0 \) and \( E(g_1^{*2}) = 1 \), then the following are equivalent
  1. \( E\|X\|^2 < \infty \) and
     \[ \sqrt{n}(\bar{X}_n - E(X)) \to Z_X \text{ weakly in } \mathcal{H} \]
  2. \( \frac{1}{\sqrt{n}} \sum_{i=1}^n g_i^* X_i(\omega) \to Z_X \text{ weakly in } \mathcal{H} \quad P - a.s. \)
Functional Data $\leftrightarrow$ Hilbert
Usual functional data

- Square-integrable functions $f : T \to \mathbb{R}$
- $T$ closed and bounded interval
- Sum and product by a scalar pointwise defined
- Metric: $d(f, g) = \int_T (f(t) - g(t))^2 \lambda(dt)$

Framework - separable Hilbert space

- $(T, \Sigma, \mu)$ measurable space so that $\mu(T) < \infty$ and $\Sigma$ with the metric $\rho(A, B) = \mu(A \Delta B)$ is separable
- $L^2(T, \mu) = \{ f : T \to \mathbb{R} \text{ measurable} \mid \int_T |f(t)|^2 \mu(dt) < \infty \}$
- Sum and product by a scalar pointwise defined
- Inner product: $\langle f, g \rangle = \int_T f(t)g(t) \mu(dt)$
- Separable Hilbert space (equivalence classes)
Linear independence
Regression models

\( \mathcal{H} \) separable Hilbert space

\[
Y = \beta_1 X_1 + \ldots + \beta_k X_k + C + \varepsilon,
\]

- \( Y \) and \( \varepsilon \), \( \mathcal{H} \)—valued random elements
- Coefficients: \( \beta_1, \ldots, \beta_k, C \in \mathcal{H} \)
- \( X_1, \ldots, X_k \) uncorrelated real random variables with \( 0 < \text{Var}(X_j) < \infty \)
- \( E(\varepsilon | (X_1, \ldots, X_k)) = E(\varepsilon) = 0 \)
- Regression function: \( m(x_1, \ldots, x_k) = C + \sum_{j=1}^{k} \beta_j x_j \)

Linear independence test

\( K = \{j_1, \ldots, j_l\} \) nonempty subset of \( \{1, \ldots, k\} \)

\[
\left\{ 
\begin{array}{c}
H_0 : \beta_j = 0 \text{ for all } j \in K \\
H_1 : \beta_j \neq 0 \text{ for any } j \in K 
\end{array}
\right.
\]
Equivalent tests

- \( E([X_j - E(X_j)][Y - E(Y)]) = \text{Var}(X_j)\beta_j, \)
- \( H0': \sum_{j \in K} \| E([X_j - E(X_j)][Y - E(Y)]) \|^2 = 0 \)

Statistic

\( \{(Y_i, X_{1i}, \ldots, X_{ki})\}_{i=1}^n, \text{iid } \mathcal{H} \bigoplus \mathbb{R}^k \)-valued random elements distributed as \( (Y, X_1, \ldots, X_k) \), where \( \bigoplus \) denotes the direct sum

\[
\hat{T}_n = \sum_{j \in K} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_{ji} - \bar{X}_j)(Y_i - \bar{Y}) \right\|^2
\]

- \( H_0 \) should be rejected at a significance level \( \alpha \in (0, 1) \) when \( \hat{T}_n > c_\alpha \)
- Aim: to find a suitable \( c_\alpha \)
Statistic: First approximation

\[ \mathcal{H}^l = \mathcal{H} \oplus \cdots \oplus \mathcal{H} \text{ is a separable Hilbert space} \]

\[ Q_i = ([X_{j1}, i - E(X_{j1})] [Y_i - E(Y)], \ldots), \]

iid \( \mathcal{H}^l \)-valued random elements for all \( i \)

- If \( E(||X_j - E(X_j)||^4) < \infty \) for all \( j \in K \), then \( 0 < Var(Q_i) < \infty \) for any \( (\beta_1, \ldots, \beta_k) \)

CLT for iid random elements in separable Hilbert spaces

\[ T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Q_i - E(Q_1)) \rightarrow Z_Q, \]

weakly in \( \mathcal{H}^l \) as \( n \rightarrow \infty \)
Statistic: Alternative expression

\[
\hat{Q}_i = \left( \left[ X_{j1,i} - \overline{X}_{j1} \right] \left[ Y_i - \overline{Y} \right], \ldots \right),
\]

\( \mathcal{H}^l \) – valued random elements for all \( i \)

\[
\hat{T}_n = \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{Q}_i \right\|^2.
\]

Asymptotic analysis

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{Q}_i = T_n + \sqrt{n} E(Q_1) + R_n,
\]

where \( R_n \) is the \( \mathcal{H}^l \)–valued random element given by

\[
R_n = \sqrt{n} \left( \left[ \overline{X}_{j1} - E(X_{j1}) \right] \left[ \overline{Y} - E(Y) \right] , \ldots \right)
\]
Theorem

Given $\alpha \in (0, 1)$, the test that rejects $H_0$ whenever $\hat{T}_n > c_\alpha$, where $c_\alpha$ is the $(1 - \alpha)$-quantile of the distribution of $\|Z_{Q}\|^2$, is asymptotically correct and consistent at a significance level $\alpha$.

Moreover, given the sequence of Pitman local alternatives given by $\beta_j^{(n)} = \frac{\delta_n}{\sqrt{n}} \beta_j$, where $\|\beta_j\| > 0$ for some $j \in K$, $\delta_n \to \infty$ and $\frac{\delta_n}{\sqrt{n}} \to 0$ as $n \to \infty$, then

$$\lim_{n \to \infty} P(T_n > c) = 1$$

for any $c > 0$.

- **Problem:** The limit distribution of $\|Z_{Q}\|^2$ is unknown
- **Solution:** Bootstrap
Exchangeable Weighted Bootstrap (I)

We will mimic the distribution of $T_n$, as its limit distribution is the same as $Z_Q$

- Let $\{W_{ni}^*\}_{i=1}^n$ an array of random variables verifying the conditions A1-5

- Initial bootstrap statistics:

$$T_n^* = \frac{1}{c\sqrt{n}} \sum_{i=1}^n W_{ni}^* Q_i - \frac{1}{c\sqrt{n}} \sum_{i=1}^n Q_i$$

- Consistency for the Exchangeable Weighted Bootstrap: $T_n^*$ converges weakly to $Z_Q$ $P - a.s.$

- **Problem:** $T_n^*$ cannot be used in practice, as it depends on unknown expected values
Exchangeable Weighted Bootstrap (II)

\[ \hat{T}_n^* = \frac{1}{c\sqrt{n}} \sum_{i=1}^{n} W_{ni}^* \hat{Q}_i - \frac{1}{c\sqrt{n}} \sum_{i=1}^{n} \hat{Q}_i \]

- Bootstrap statistic: \( \hat{T}_n^* = \| \hat{T}_n^* \|^2 \)

Consistency

Let \( \hat{X}_i = (X_{j_1,i}, \ldots, X_{j_l,i}) \in \mathbb{R}^l \) \((i \in \{1, \ldots, n\})\)

\[ \hat{T}_n^* = T_n^* + (E(Y) - \overline{Y}) R_{n}^{X^*} + (E(\hat{X}) - \overline{\hat{X}}) R_{n}^{Y^*}, \]

with

\[ R_{n}^{X^*} = \frac{1}{c\sqrt{n}} \sum_{i=1}^{n} W_{ni}^* \hat{X}_i - \frac{1}{c\sqrt{n}} \sum_{i=1}^{n} \hat{X}_i, \]

\[ R_{n}^{Y^*} = \frac{1}{c\sqrt{n}} \sum_{i=1}^{n} W_{ni}^* Y_i - \frac{1}{c\sqrt{n}} \sum_{i=1}^{n} Y_i. \]
Wild Bootstrap (I)

We will mimic the distribution of $T_n$, as its limit distribution is the same as $Z_Q$

- Let $\{\xi_i^*\}_{i=1}^n$ be a sequence of iid random variables with mean 0, variance 1 and satisfying that
  \[
  \int_0^\infty (P(|\xi_1^*| > t)^{1/2})dt < \infty
  \]

- Initial bootstrap statistic:
  \[
  S_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i^* Q_i
  \]

- Consistency for the Wild Bootstrap: $S_n^*$ converges weakly to $Z_Q$ $P - a.s.$

- **Problem**: $S_n^*$ cannot be used in practice, as it depends on unknown expected values
Wild Bootstrap (II)

$$\hat{S}_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^* \hat{Q}_i$$

- Bootstrap statistic: $$\hat{S}_n^* = ||\hat{S}_n^*||^2$$

Consistency

$$\hat{S}_n^* = S_n^* + (E(Y) - \bar{Y})R_{n}^{X^*} + (E(\bar{X}) - \bar{\bar{X}})R_{n}^{Y^*} + (E(Y) - \bar{Y})(E(\bar{X}) - \bar{\bar{X}})R_{n}^{\xi^*},$$

con

$$R_{n}^{X^*} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^* \bar{X}_i, \quad R_{n}^{Y^*} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^* Y_i$$

$$R_{n}^{\xi^*} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^*$$
Theorem - Exchangeable Weighted Bootstrap

Let $\alpha \in (0, 1)$, the test that rejects $H_0$ whenever $\hat{T}_n > c_{\alpha}^n$, where $c_{\alpha}^n$ is the $(1 - \alpha)$-quantile of the distribution of $\hat{T}_n$, is asymptotically correct and consistent $P$-a.s. at the significance level $\alpha$.

Theorem - Wild Bootstrap

Let $\alpha \in (0, 1)$, the test that rejects $H_0$ whenever $\hat{T}_n > c_{\alpha}^n$, where $c_{\alpha}^n$ is the $(1 - \alpha)$-quantile of the distribution of $\hat{S}_n^\star$, is asymptotically correct and consistent $P$-a.s. at the significance level $\alpha$.

In practice

- Direct application for functional data
- Bootstrap distribution approximated by MonteCarlo
  - Random “weights” according to the corresponding distributions
Fuzzy ↔ Hilbert
Likert scale

- When the experimental information cannot be quantified by means of precise numbers, nominal or ordinal scales are frequently used
  - Perceptions or subjective assessments
  - Example: Likert scale

- Drawbacks
  - The available statistical tools are limited
  - The transition between the categories cannot be gradual
  - The categories are not equally perceived by different observers $\Rightarrow$ precision and variability are not well-captured
Alternative to the Likert scales

- **Grading scale** between the worst case and the best case (in %) to chose a single number
- **Drawback**
  - ✓ Indecision facing values virtually indistinguishable
- **Fuzzy scale**
  - ✓ generalization of the characteristic function of a set $A$, which takes values in the whole interval $[0, 1]$, instead of only 0 or 1
  - ✓ the value $f(x) \in [0, 1]$ that takes on the fuzzy set $f$ for a given $x \in A$ represents the membership degree of $x$ to the set $A$
  - ✓ “gray-scale” or opinion or assessments
Advantages of the fuzzy scale

- **Imprecise perceptions** can be expressed in a more complete way than by using traditional scales
- The metric and arithmetic structures of the space of fuzzy sets make it possible to handle fuzzy data in a relatively easy theoretical framework

Drawback of the fuzzy scale

- **Complex data**: functional values

Is it worth it?

- It depends on the level of detail that the practitioners need for their data
## Experiment “Perceptions”

### Analizing Perceptions

<table>
<thead>
<tr>
<th>Linguistic Descriptor</th>
<th>Next Trial</th>
<th>Min</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
<th>60%</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large</td>
<td></td>
<td>10%</td>
<td>20%</td>
<td>30%</td>
<td>40%</td>
<td>50%</td>
<td>60%</td>
<td>70%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Perception about the relative length**

- **Trial Number:** 1

---

**Motivation**

**Fuzzy ↔ Hilbert**

**FDA ↔ Hilbert**

**Linear independence**

**Test**

**Asymptotics**

**Bootstrap**

**ANOVA**

**ANOVA in Hilbert spaces**

**ANOVA for RFS**
Experiment’s details

- 8 woman and 9 men with different profiles
- Each person made 3 trials for 9 sizes (3x9)
- Equally-spaced sizes on the support [0,100]
- Random position and location within each sequence and among sequences

Experiment’s objective

To determine whether there are significant differences among the individual’s perceptions and between man and women

- In order to compare the overall perception, we consider the average as summary measure
- Since all the participants assessed the same set of sizes, the mean values should be the same but for differences in perception
- Differences for each one of the 9 sizes with be investigated as well
Methodological objectives

- To formalize the connection between the space of fuzzy sets and a certain cone of a Hilbert space
- To develop a bootstrap test for the equality of means of $k$ Hilbert-valued random variables based on the variance decomposition
- To show theoretically the correction, consistency and power under local alternatives
- To particularize to the fuzzy case and show some empirical results
Fuzzy sets

\[ \mathcal{F}_c(\mathbb{R}^p) = \{ U : \mathbb{R}^p \rightarrow [0, 1] \mid U_\alpha \in \mathcal{K}_c(\mathbb{R}^p) \quad \forall \alpha \in (0, 1] \} \]

- \( U_\alpha = \{ x \in \mathbb{R}^p \mid U(x) \geq \alpha \} \) if \( \alpha > 0 \),
- \( \mathcal{K}_c(\mathbb{R}^p) = \{ A \subset \mathbb{R}^p \mid A \neq \emptyset \text{ compact and convex} \} \)

Arithmetic for \( U, V \in \mathcal{F}_c(\mathbb{R}^p) \) and \( \gamma \in \mathbb{R} \)

\( (U + \gamma V)_\alpha = \{ u + \gamma v \mid u \in U_\alpha, v \in V_\alpha \} \) with \( \alpha \in (0, 1] \)

Support function of \( A \in \mathcal{F}_c(\mathbb{R}^p) \) (Puri & Ralescu, 1985)

- \( s_A : \mathbb{S}^{p-1} \times (0, 1] \rightarrow \mathbb{R} \) with \( \mathbb{S}^{p-1} \) unit sphere on \( \mathbb{R}^p \)
  \[ s_A(u, \alpha) = \sup_{a \in A_\alpha} \langle u, a \rangle \text{ for all } u \in \mathbb{S}^{p-1}, \alpha \in (0, 1] \]
  \( \langle \cdot, \cdot \rangle = \text{inner product in } \mathbb{R}^p \)
- \( s \) preserves the semilinear structure
  \[ s(A + \gamma B) = s(A) + \gamma s(B) \text{ with } A, B \in \mathcal{F}_c(\mathbb{R}^p), \gamma \in \mathbb{R}^+ \]
Functionality of the space $\mathcal{H} = L^2(S^{p-1} \times (0, 1], \theta_p \times \lambda)$

- $\theta_p$ normalized Lebesgue measure on $S^{p-1}$
- $\lambda$ Lebesgue measure on $(0, 1]$

Mid/spr-decomposition of $f \in \mathcal{H}$

\[
\text{mid } f(u, \alpha) = \left( f(u, \alpha) - f(-u, \alpha) \right)/2 \\
\text{spr } f(u, \alpha) = \left( f(u, \alpha) + f(-u, \alpha) \right)/2
\]

$\Rightarrow f = \text{mid } f + \text{spr } f$ with $\text{mid } f$ even and $\text{spr } f$ odd

Scalar product

Let $\theta > 0$ and $\varphi$ be an $L^2$ density with support $[0, 1]$

\[
\langle f, g \rangle_{\theta, \varphi} = [\text{mid } f, \text{mid } g]_\varphi + \theta [\text{spr } f, \text{spr } g]_\varphi
\]

\[
[f, g]_\varphi = \int_{(0, 1]} \int_{S^{p-1}} f(u, \alpha)g(u, \alpha)d\theta_p(u)\varphi(\alpha)d\alpha
\]

$\{ \mathcal{H}, \langle \cdot, \cdot \rangle_{\theta, \varphi} \}$ is a separable Hilbert space
Orthogonal decomposition of the support function of $A \in \mathcal{F}^2(\mathbb{R}^p)$

- The projection of $A_\alpha$ over the direction $u$ is the interval
  \[ \Pi_u A_\alpha = [-s_A(-u, \alpha), s_A(u, \alpha)] \]
  \[ \checkmark \quad \text{mid } s_A(u, \alpha) = \text{mid } \Pi_u A_\alpha, \quad \text{spr } s_A(u, \alpha) = \text{spr } \Pi_u A_\alpha \]
  \[ \Rightarrow s_A = \text{mid } A + \text{spr } A \]

Class of distances in $\mathcal{F}^2_c(\mathbb{R}^p)$ (Trutschnig et al., 2009)

\[ (D_{\theta,\varphi}(A, B))^2 = \langle s_A - s_B, s_A - s_B \rangle_{\theta,\varphi} = \|s_A - s_B\|^2_{\theta,\varphi} \]

for all $A, B \in \mathcal{F}^2_c(\mathbb{R}^p) = \{ A \in \mathcal{F}_c(\mathbb{R}^p) \mid s_A \in \mathcal{H} \}$

- $(\mathcal{F}^2_c(\mathbb{R}^p), D_{\theta,\varphi})$ is isometric to a convex and closed cone of $\mathcal{H}$

- Alternative expression: given $\tau \in (0, 1)$
  \[ (D_{\tau}(U, V))^2 = (1 - \tau) (\|\text{mid } U - \text{mid } V\|)^2 + \tau (\|\text{spr } U - \text{spr } V\|)^2 \]
Random Fuzzy Set - RFS (Puri & Ralescu, 1986)

- Let \((\Omega, \mathcal{A}, P)\) be a probability space
- \(\mathcal{X} : \Omega \to \mathcal{F}_c^2(\mathbb{R}^p)\) is a RFS if and only if \(\mathcal{X}_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R}^p)\) are compact random sets \(\forall \alpha \in (0, 1]\) or, equivalently,
  - \(\mathcal{X}\) is Borel measurable wrt to the metric \(D_{\theta, \varphi}\)
  - \(s\mathcal{X} : \Omega \to \mathcal{H}\) is an \(\mathcal{H}\)-random element

Expected value of an RFS (Puri & Ralescu, 1986)

If \(s\mathcal{X} \in L^1(\Omega, \mathcal{A}, P)\), then the expected value of \(\mathcal{X}\) is the unique fuzzy set \(E(\mathcal{X}) \in \mathcal{F}_c^2(\mathbb{R}^p)\) with \(sE(\mathcal{X}) = E(s\mathcal{X})\)

Variance of a RFS (Körner & Näther, 2002)

If \(\|s\mathcal{X}\|_{\theta, \varphi}^2 \in L^1(\Omega, \mathcal{A}, P)\), the \(D_{\theta, \varphi}\)-variance of \(\mathcal{X}\) is

\[
\sigma_X^2 = E\left(\left[D_{\theta, \varphi}(\mathcal{X}, E[\mathcal{X}])\right]^2\right) = E\|s\mathcal{X} - sE(\mathcal{X})\|_{\theta, \varphi}^2 = \sigma_{s\mathcal{X}}^2
\]
ANOVA
The problem

- $\mathcal{H}$ = separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$
- $H_1, \ldots, H_k$ are $k$ $\mathcal{H}$-valued random elements with means $m_1, \ldots, m_k$, and covariance functions $K_1, \ldots, K_k$
- **Aim**: To test
  - $H_0 : m_1 = \ldots = m_k$ against
  - $H_a : \exists i_1 \neq i_2$ with $m_{i_1} \neq m_{i_2}$


- Asymptotic ANOVA test for functional data based on a statistics measuring the sum of the pairwise differences between the sample means
The statistics (I)

- Let \( \{H_{ij}\}_{j=1}^{n_i} \) be a random sample obtained from \( H_i \) for all \( i \in \{1 \ldots, k\} \) and \( n = n_1 + \ldots + n_k \).

- **Natural extension** of the ANOVA test statistic:

\[
A_n = \sum_{i=1}^{k} n_i \left\| \overline{H_i} - \overline{H} \right\|^2,
\]

where \( \overline{H_i} = \sum_{j=1}^{n_i} H_{ij} / n_i \) and \( \overline{H} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} H_{ij} / n \).

- **Decomposition:**

\[
A_n = \sum_{i=1}^{k} n_i \left\| \overline{H_i^c} - \overline{H} \right\|^2 + \sum_{i=1}^{k} n_i \| m_i - \mu_n \|^2
\]

\[
+ 2 \sum_{i=1}^{k} n_i \left\langle \overline{H_i^c} - \overline{H}^c, m_i - \mu_n \right\rangle,
\]

where \( H_{ij}^c = H_{ij} - m_i \) and \( \mu_n = \frac{1}{n} \sum_{i=1}^{k} n_i m_i \).
The statistics (II)

- The first term can be expressed as follows:

\[
\sum_{i=1}^{k} n_i \left\| H_{i.}^c - \overline{H}^c \right\|^2 = \eta_n (\sqrt{n_1} \cdot \overline{H}_{1.}^c, \ldots, \sqrt{n_k} \cdot \overline{H}_{k.}^c),
\]

where \( \eta_n : \mathcal{H} \times \ldots \times \mathcal{H} \to \mathbb{R} \) associates \((h_1, \ldots, h_k)\) with

\[
\sum_{i=1}^{k} \left\| h_i - \sum_{l=1}^{k} \alpha_{li}^n h_l \right\|^2,
\]

where \( \alpha_{li}^n = \sqrt{n_l/n_i} / \sum_{r=1}^{k} (n_r/n_i) \)
Asymptotic distribution

If \( n_i \to \infty \) and \( n_i/n \to p_i > 0 \) when \( n \to \infty \) for all \( i \in \{1, \ldots, k\} \), then under \( H_0 \), \( A_n \) converges in distribution to

\[
A = \sum_{i=1}^{k} \left\| Z_i - \sum_{l=1}^{k} \alpha_{li} Z_l \right\|^2,
\]

where \( \alpha_{li} = \sqrt{p_l/p_i} / \sum_{r=1}^{k} (p_r/p_i) \) for all \( i, l \in \{1, \ldots, k\} \), and \( Z_1, \ldots, Z_k \) are centered Gaussian processes on \( \mathcal{H} \) with covariance functions \( K_1, \ldots, K_k \).

Tools to get the result

- CLT in Hilbert space
- Slutsky theorem
- Continuous mapping
Local alternatives

- \( m_i = m^* + \frac{\delta_n}{\sqrt{n}} m_i^* \) with \( m^*, m_i^* \in \mathcal{H} \), for \( \delta_n \in \mathbb{R}^+ \) and \( n \in \mathbb{N} \), and so that there exists \( i_1 \neq i_2 \) with \( m_{i_1}^* \neq m_{i_2}^* \)
- If \( \delta_n/\sqrt{n} \to 0 \), then \( \| m_i - m_j \|^2 \to 0 \) as \( n \to \infty \) for all \( i, j \in \{1, \ldots, k\} \)
- Although \( \mathcal{H}_0 \) is not satisfied, the alternative approach it with ratio \( \delta_n/\sqrt{n} \) for all \( n \in \mathbb{N} \)

Result

If \( n_i \to \infty \) and \( n_i/n \to p_i > 0 \) when \( n \to \infty \) for all \( i \in \{1, \ldots, k\} \), \( \delta_n \to \infty \) and \( \delta_n/\sqrt{n} \to 0 \) as \( n \to \infty \), the \( P(A_n \leq t) \to 0 \) as \( n \to \infty \) for all \( t \in \mathbb{R} \)
Studentized statistic

If \( n_i \to \infty \) and \( n_i/n \to p_i > 0 \) and \( H_i \) is not degenerated for some \( i \), then

\[
B_n = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \| H_{ij} - \overline{H}_i \|^2 \to \sum_{i=1}^{k} p_i E \| H_i - m_i \|^2 > 0 \text{ c.s. } [P].
\]

Consequently, if \( H_0 \) is true,

\[
T_n = \frac{1}{k} \sum_{i=1}^{k} n_i \| \overline{H}_i - \overline{H} \|^2 \to T = \frac{\sum_{i=1}^{k} \| Z_i - \sum_{l=1}^{k} \alpha_{li} Z_l \|^2}{\sum_{i=1}^{k} p_i E \| H_i - m_i \|^2}
\]

Under local alternatives, if \( \delta_n \to \infty \) as \( n \to \infty \), then

\[
P(A_n/B_n \leq t) \to 0 \text{ as } n \to \infty \text{ for all } t \in \mathbb{R}
\]
Bootstrap approximation
(naive bootstrap in separable Hilbert spaces)

- If $H_{ij}^*$ are $n_i$ random elements picked from
  \[ \{H_{i1} - H_i, \ldots, H_{in_i} - H_i\} \] at random \( \Rightarrow \)
  \[ \sqrt{n_i \cdot H_i^*} \to Z_i \text{ in law } [P] - a.s. \text{ as } n_i \to \infty \]

- Bootstrap statistic:
  \[
  A_n^* = \sum_{i=1}^{k} n_i \left\| H_i^* - \bar{H}_i^* \right\|^2 = \sum_{i=1}^{k} n_i \left\| H_i^* - \sum_{l=1}^{k} \alpha_{ili} H_i^* \right\|^2,
  \]
  \( \Rightarrow A_n^* \to A \text{ in law } [P] - a.s. \)

- Bootstrap statistic:
  \[
  B_n^* = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left\| H_{ij}^* - \bar{H}_i^* \right\|^2 \to \sum_{i=1}^{k} p_i E\left\| H_i - m_i \right\|^2 > 0
  \]
  in probability \( [P] - a.s. \)
ANOVA test for fuzzy data: The problem

- Let \( \mathcal{X}_1, \ldots, \mathcal{X}_k \) be \( k \) independent \( \mathcal{F}^2_c(\mathbb{R}^p) \)-valued RFS, with means \( E(\mathcal{X}_i) = m_i \in \mathcal{F}^2_c(\mathbb{R}^p) \), variances \( \sigma^2_{\mathcal{X}_i} \), and being \( K_i \) the covariance function of \( s_{\mathcal{X}_i} \).

- **Objective**: To test

  \[
  H_0 : m_1 = \ldots = m_k \quad \text{against} \quad H_a : \exists i_1 \neq i_2 \text{ with } m_{i_1} \neq m_{i_2},
  \]

  from a random sample \( \{\mathcal{X}_{ij}\}_{j=1}^{n_i} \) drawn from \( \mathcal{X}_i \).

**Theorem (I)**

If \( n_i \to \infty, n_i/n \to p_i > 0 \) as \( n \to \infty \) for all \( i \in \{1, \ldots, k\} \) and \( \mathcal{X}_i \) is non-degenerated for some \( i \), then if \( H_0 \) is true, the following convergence is law is satisfied:
Theorem (II)

\[ T_n = \sum_{i=1}^{k} n_i \left( D_{\theta}^{\varphi}(\overline{X}_i., \overline{X}_.) \right)^2 \]

\[ \Rightarrow \mathcal{T} = \sum_{i=1}^{k} \left( \sqrt{\sum_{l=1}^{k} \alpha_{li} Z_l} \right)^2 \]

where \( Z_1, \ldots, Z_k \) are Gaussian processes in \( \mathcal{L}^2(\mathbb{S}^{p-1} \times (0, 1], \lambda_p \times \lambda) \) with covariance functions \( K_1, \ldots, K_k \)

**Under local alternatives**, if \( \delta_n \to \infty \) as \( n \to \infty \), then
\[ P(T_n \leq t) \to 0 \text{ as } n \to \infty \text{ for all } t \in \mathbb{R} \]

Let \( \{ \mathcal{X}_{ij}^* \}_{j=1}^{n_i} \) a random sampled drawn from \( \{ \mathcal{X}_{i1}, \ldots, \mathcal{X}_{in_i} \} \), then
\[ T_n^* = \sum_{i=1}^{k} n_i \left( D_{\theta}^{\varphi}(\overline{X}_i^*, \overline{X}_., \overline{X}_i^* + \overline{X}.) \right)^2 \]

\[ \Rightarrow \mathcal{T} \quad [P] - a.s. \]

\[ \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( D_{\theta}^{\varphi}(\mathcal{X}_{ij}^*, \overline{X}_i.) \right)^2 \]
Behren-Fisher problem: correction

- When the Behren-Fisher is evident, better results are usually obtained by correcting the denominator as follows:

\[ T'_n = \frac{\sum_{i=1}^{k} n_i \left( D^\varphi_\theta (X_{i..}, \bar{X}..) \right)^2}{\sum_{i=1}^{k} \frac{1}{n_i^2} \sum_{j=1}^{n_i} \left( D^\varphi_\theta (X_{ij}, \bar{X}_{i..}) \right)^2} \]

- The analogous bootstrap version can be considered to approximate the sampling distribution under \( H_0 \)

- The correctness and consistency of the test can be proved analogously
Simulations: Empirical size

- $k = 3$ independent RFS with **triangular** values have been simulated parametrized by 3 v.a. $(C, L, R)$
- $C \sim N(0, 1)$ $L R \sim \chi^2$ independent
- The selected metric is $D_{1/3}^\lambda$
- There is no relevant differences with other distributions or dependence structures
- For completeness, we compare with the ANOVA applied to the **defuzzified values** obtained by the 0.5-average

<table>
<thead>
<tr>
<th>$(n_1, n_2, n_3)$</th>
<th>(30,30,30)</th>
<th>(30,30,100)</th>
<th>(30,100,100)</th>
<th>(100,100,100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ (fuzzy)</td>
<td>5.11</td>
<td>4.74</td>
<td>4.49</td>
<td>4.97</td>
</tr>
<tr>
<td>$A_2$ (real)</td>
<td>5.09</td>
<td>4.69</td>
<td>4.48</td>
<td>5.01</td>
</tr>
</tbody>
</table>
Simulations: power

- From a situation where the ANOVA for RFD and their defuzzified values have the same size
- Balance design with 30 data per variable
- $H_0$ is perturbed by fuzzy values as follows
  \[ d_0 : \mu_1 = \mu_2 = \mu_3, \]
  \[ d_i : \mu_1 = \mu_2 \text{ y } \mu_3 = \mu_1 + 0.4i \mu_1 \]

<table>
<thead>
<tr>
<th></th>
<th>$d_0$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ (fuzzy)</td>
<td>4.77</td>
<td>6.77</td>
<td>13.18</td>
<td>29.94</td>
<td>57.24</td>
<td>88.99</td>
<td>99.85</td>
</tr>
<tr>
<td>$A_2$ (real)</td>
<td>4.77</td>
<td>5.99</td>
<td>8.96</td>
<td>15.28</td>
<td>24.30</td>
<td>37.28</td>
<td>50.81</td>
</tr>
</tbody>
</table>

- Loss of power due to the simplification
• **Dotted and dashed lines**: Two fuzzy data for line segments of 26.76% and 96.27% relatives chose by two participants (left?female, right?male)

• **Thick line**: average of the 27 trials of both participants
Distances between the data represented in the previous figure

- **Dotted line**: woman
- **Dashed line**: man
- **Thick line**: Averages of both individuals
Performed ANOVA tests

- **Differences among individuals**
  
  $k = 17$ samples of sizes $n_i = 27$

- **Differences between sex**
  
  $k = 2$ with $n_1 = 27 \cdot 8 = 216$ and $n_2 = 27 \cdot 9 = 243$

- **Differences among women**
  
  $k = 8$ with $n_i = 27$

- **Differences among men** $k = 9$ with $n_i = 27$

- **Differences between sex for each one of the 9 sizes**
  
  $k = 2$ with $n_1 = 3 \cdot 8 = 24$ and $n_2 = 3 \cdot 9 = 27$

  ✓ $p$-values approximated by bootstrap with $B = 10,000$ replications
Sample means for the relative lengths of 26.76%, 96.27% and the average of the 9 lengths

- **Dotted line:** women
- **Dashed line:** men
Dependence on $\tau(1)$

- The results depend on $D_\tau$, and then, on $\tau$
- The importance of the spreads against the mid-points should depend on the *relative variability* $\Rightarrow$ the $p$-values will be computed in terms of

$$q = \frac{\tau \Var(spr \mathcal{X})}{(1 - \tau)\Var(mid \mathcal{X}) + \tau \Var(spr \mathcal{X})}.$$  

- $q$ is an increasing function of $\tau$ and verifies that
  - $q \to 0^+ \iff \tau \to 0^+$
  - $q = .5 \iff (1 - \tau)\Var(mid \mathcal{X}) = \tau \Var(spr \mathcal{X})$
  - $q \to 1^- \iff \tau \to 1^-$
### Dependence on $\tau$ (II)

- In practice, $\text{Var}(\text{mid } \mathcal{X})$ is usually much higher than $\text{Var}(\text{spr } \mathcal{X})$, due to the difference of magnitudes $\Rightarrow$ weights $\tau \leq 0.5$ are usually associates with very small values for $\rho$.

- The population variances in the definition of $\rho$ have been approximated by the analogous estimators based on the total sample for each case.
Results: individuals and sex

$p$-values of the ANOVA test for individuals ($k = 17$), and sex ($k = 2$) as a function of $\varrho$
Results: males and females

\[ p\text{-values of the ANOVA test for women (}k = 8\text{) and men (}k = 7\text{) as a function of } \varrho \]
Results for $\rho = 0.05$ and $\rho = 0.25$

<table>
<thead>
<tr>
<th></th>
<th>Individuals</th>
<th>Sex</th>
<th>Men</th>
<th>Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0.05$</td>
<td>0.734</td>
<td>0.111</td>
<td>0.979</td>
<td>0.384</td>
</tr>
<tr>
<td>$\rho = 0.25$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.165</td>
<td>0.000</td>
</tr>
</tbody>
</table>

- Comparison of the result for sex with the corresponding to the data of the Likert scale

<table>
<thead>
<tr>
<th></th>
<th>VERY</th>
<th>SMALL</th>
<th>SMALL</th>
<th>MEDIUM</th>
<th>BIG</th>
<th>BIG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td></td>
<td>18.9%</td>
<td>20.2%</td>
<td>24.3%</td>
<td>22.2%</td>
<td>14.4%</td>
</tr>
<tr>
<td>Women</td>
<td></td>
<td>12.7%</td>
<td>25.9%</td>
<td>21.3%</td>
<td>20.3%</td>
<td>18.5%</td>
</tr>
</tbody>
</table>

- $p$-value of the test $\chi^2$: 0.204
- $p$-value of the ANOVA bootstrap test: 0.271
Conclusions

- **Main tool:** Framework to use in practice key previous results to prove the validity of bootstrap methods
  - ✓ The previous results were for the sample mean in the context of empirical process
  - ✓ Analogous results in separable Hilbert spaces can be derived by linking properly both contexts

- **Impact:** many statistics can be written in terms of means in the suitable spaces
  - ✓ Test for the linear independence with application to FDA
  - ✓ ANOVA test with application to RFS